Recitation 5

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Problems

Problem 1. W is not a subspace of $\mathbb{R}^3$ since it's not closed under addition: take vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$. They are in the set W, but their sum $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is no longer in W.

Problem 2. This set V is exactly the null space of the matrix $A = [1\ 1\ 1]$. Indeed, A defines a linear transformation $A\colon \mathbb{R}^3 \to \mathbb{R}$ sending a vector $x = \begin{bmatrix}a\\b\\c\end{bmatrix}$ to $Ax = a + b + c$. By Theorem 2 in Section 4.2 of the textbook, null space of any matrix is a subspace. Alternatively, you can just directly check that V satisfies the two axioms of being a subspace.

Problem 3. Put the vectors into a 4×4 matrix A, do row reduction. You will see there are only 3 pivotal columns (and rows). **Not every column is pivotal** $\Rightarrow$ vectors are **dependent**. **Not every row** is pivotal, so the vectors **do not span** $\mathbb{R}^4$.
The first three columns are pivotal, so $\{v_1, v_2, v_3\}$ for a basis of $Col(A) = Span(v_1, v_2, v_3, v_4)$.

Problem 4. Row reduce the matrix. There are three pivots. So **every row and every column** is pivotal. Therefore columns do form a basis of $\mathbb{R}^3$, and $Col(A) = \mathbb{R}^3$. Matrix A is invertible, and its inverse is $A^{-1} = \begin{bmatrix}0 & 0 & 1\\0.6 & 0.4 & -0.4\\-0.2 & -0.3 & 0.8\end{bmatrix}$

Problem 5. Row reduce, the first two columns are pivotal, so they for a basis for $Col(A)$. So $\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\7\end{bmatrix}\}$ is a basis of $Col(A)$. To find a basis for $Nul(A)$, we need to solve system of equations $Ax = 0$. Row reducing, x_3 is a free variable, $x_2 = -4x_3$ and $x_1 = 9x_3$. So the general solution is of the form $\begin{bmatrix}9x_3\\-4x_3\\x_3\end{bmatrix} = x_3\begin{bmatrix}9\\-4\\1\end{bmatrix}$. So the vector $\begin{bmatrix}9\\-4\\1\end{bmatrix}$ is a basis for $Nul(A)$.

Problem 6. Vector $[1] \in \mathbb{R}$ is a basis for $Col(A)$. Solving the system $Ax = 0$, variables x_2, x_3 are free, and $x_1 = -3x_2 + x_3$. So general solution is the form $\begin{bmatrix}-3x_2 + x_3\\x_2\\x_3\end{bmatrix} = x_2\begin{bmatrix}-3\\1\\0\end{bmatrix} + x_3\begin{bmatrix}1\\0\\1\end{bmatrix}$. So vectors $\begin{bmatrix}-3\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ form a basis for $Nul(A)$.

Problem 7. Put vectors into matrix, row reduce, there are two pivots. Every column and row is pivotal so the two vectors form a basis of $\mathbb{R}^2$.
To find coordinates of $x = \begin{bmatrix}2\\0\end{bmatrix}$ we need to solve the system $\begin{bmatrix}-1 & 1\\0 & 2\end{bmatrix} y = \begin{bmatrix}2\\0\end{bmatrix}$. The solution is $y = \begin{bmatrix}2\\-1\end{bmatrix}$.

Problem 8. Since $V = Nul(A)$ for $A = [1\ 1\ 1]$ we are again looking for a basis of null-space. Solving $Ax = 0$ we find a basis. For example, $\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$ will form a basis.

To find the coordinates $[x]_{\mathcal{B}}$ of the vector $x = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$ is this basis, we need to solve the system $\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} y = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$. The solution is $y = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and has two entries, which shouldn't be surprising, since there are two vectors in the basis.

Problem 9. The transformation T sends a polynomial $ax^3 + bx^2 + cx + d$ to the polynomial $3ax^2 + 2bx + c$. The latter polynomial is zero if and only if $a = b = c = 0$. So the kernel of T consists of all polynomials $ax^3 + bx^2 + cx + d$ having $a = b = c = 0$, i.e. $\ker(T) = \{d, d \in \mathbb{R}\} = Span(1)$ is the space of constant polynomials.

Problem 10.

1. False.
2. False.
3. True.
4. True.
5. False.
6. False.

Problem 11. Expressions $b = x_1v_1 + x_2v_2 + x_3v_3$ correspond to solutions (x_1, x_2, x_3) of the system of equations $Ax = b$ with $A = (v_1|v_2|v_3)$. Since v_1, v_2, v_3 are dependent, not every column of A is pivotal, and so there are free variables (we know the system $Ax = b$ is consistent since $b \in Span(v_1, v_2, v_3)$ by assumption). Free variables imply non-unique solution. So b can be expressed as $b = x_1v_1 + x_2v_2 + x_3v_3$ in more than one way.